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Entanglement Teleportation Via a Two-qubit Heisenberg Chain under a Nonuniform Magnetic Field

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Abstract By introducing the nonuniform magnetic field, we investigate the entanglement teleportation via two-qubit Heisenberg chain. We show that for ferromagnetic chain, the opposite direction magnetic field on the two-qubit chain can excite the teleported entanglement C_{out}, while the uniform magnetic field can not do it. The effect of the uniform magnetic field B and the nonuniform magnetic field b on the threshold temperature T_c is also plotted. Our study on the average fidelity of this quantum channel system shows that the magnetic field in opposite direction can result in the ideal average fidelity no matter whether the chain is ferromagnetic or antiferromagnetic.

Keywords Entanglement teleportation · Heisenberg chain · Nonuniform magnetic field

1 Introduction

As a valuable resource in quantum information and quantum computation [1, 2], quantum entanglement has attracted numerous attention over past decade years. One of the most fascinating features about quantum entanglement is the nonlocal correlation property which does not exist classically. This nonlocal property enables a striking phenomenon called quantum teleportation which has been extensively studied both experimentally and theoretically in the past few years [3–5]. The teleportation through some solid systems such as quantum chains is an important emerging field [6–8]. A quantum chain also referred as spin chain is a one-dimensional array of qubits which are coupled permanently by mutual interaction, which can be used to teleport a quantum state. Recently, thermally entangled state of a two-qubit Heisenberg chain has been considered as a quantum channel in many papers [9–11]. Ref. [10] studies the effects of spin orbit coupling on entanglement teleportation through a two-qubit spin chain. In Ref. [12] the author studied the influence of magnetic field and anisotropy on quantum teleportation via a Heisenberg XY chain, but there the magnetic field

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is uniformly distributed. In papers [13–15] the authors found that there are many nontrivial features about the thermal entanglement of a Heisenberg chain under a nonuniform magnetic field. However, the teleportation via a two-qubit chain under a nonuniform magnetic field has not been discussed.

Therefore, in this paper we investigate the influence of the nonuniform magnetic field on the entanglement teleportation through a two-qubit Heisenberg chain. We consider the input state is a pure two-qubit entangled state. This paper is organized as follows. In Sect. 2, we give the Hamiltonian of our channel system and (briefly review a measure of entanglement, the concurrence) deduce the output entanglement and the average fidelity in the process of entanglement teleportation. We conclude our results in Sect. 3.

2 Theoretical Treatment and Results

The channel Hamiltonian of a two-qubit isotropic Heisenberg chain in a external uniform magnetic field B and a nonuniform magnetic field b along the Z-axis is

$$H = \frac{J}{2}(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y + \sigma_1^z\sigma_2^z) + \frac{(B+b)}{2}\sigma_1^z + \frac{(B-b)}{2}\sigma_2^z \tag{1}$$

here $(\sigma_i^x, \sigma_i^y, \sigma_i^z)$ are the vector of Pauli matrices, and J is the real coupling coefficient. The parameter $J > 0$ means that the chain is antiferromagnetic, and ferromagnetic for $J < 0$. The magnetic field on the two-qubit are $B + b$ and $B - b$, respectively, the value of b controls the degree of inhomogeneity.

For a spin system in equilibrium at temperature T, the density matrix is $\rho = (1/Z) \times \exp(-H/k_B T)$, where H is the Hamiltonian of this system, Z is the partition function and k_B is the Boltzmann constant. Usually we write $k_B = 1$. The entanglement of two-qubit can be measured by the concurrence C [16], which is defined as $C = \max(0, 2\max\lambda_i - \sum_{i=1}^4 \lambda_i)$ where λ_i is the square roots of the eigenvalues of the matrix $R = \rho(\sigma_1^y \otimes \sigma_2^y)\rho^*(\sigma_1^y \otimes \sigma_2^y)$ where the asterisk indicates complex conjugation. The concurrence C ranges from zero for separable states to one for maximally entangled states and is available no matter whether ρ is pure or mixed.

Without loss of generality, in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the eigenvalues and the eigenvectors of the channel are easily obtained as following forms $H|\psi^\pm\rangle = (\frac{J}{2} \pm B)\psi^\pm$ and $H|\Sigma^\pm\rangle = (-\frac{J}{2} \pm J\eta)|\Sigma^\pm\rangle$, and the eigenstates are $|\psi^+\rangle = |00\rangle, |\psi^-\rangle = |11\rangle$ and $\Sigma^\pm = N^\pm[(\xi \pm \eta)|01\rangle + |10\rangle]$ where $\xi = b/J$ and $\eta = \sqrt{1+\xi^2}$. The normalization constants are $N^\pm = 1/\sqrt{2\eta(\eta \pm \xi)}$. Then we can calculate the density matrix of this channel system in equilibrium (temperature T). It can be written as the following form

$$\rho(T) = \begin{pmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \mu_+ & y & 0 \\ 0 & y & \mu_- & 0 \\ 0 & 0 & 0 & \omega_2 \end{pmatrix}.$$

The exact values of these nonzero matrix elements can be obtained by knowing the spectrum of H. We obtain

$$\omega_1 = \frac{1}{Z}e^{-\beta(\frac{J}{2}+B)}, \omega_2 = \frac{1}{Z}e^{-\beta(\frac{J}{2}-B)},$$

$$\begin{aligned}\mu_{+}&=\frac{1}{Z}e^{\frac{J}{2}\beta}\left[\cosh(J\beta\eta)-\frac{\xi}{\eta}\sinh(J\beta\eta)\right],\\ \mu_{-}&=\frac{1}{Z}e^{\frac{J}{2}\beta}\left[\cosh(J\beta\eta)+\frac{\xi}{\eta}\sinh(J\beta\eta)\right],\\ y&=-\frac{1}{Z\eta}e^{\frac{J}{2}\beta}\sinh(J\beta\eta),\end{aligned}\tag{2}$$

where the partition function Z is given by

$$Z=2e^{-\frac{J}{2}\beta}\cosh(\beta B)+2e^{\frac{J}{2}\beta}\cosh(J\beta\eta).\tag{3}$$

Based on knowing the density matrix of the channel system, we consider the entanglement teleportation through it. The standard teleportation through an entangled mixed state resource can be regarded as a general depolarizing channel [17, 18]. Similar to the standard teleportation protocol, the entanglement teleportation for the mixed channel of an input entangled state is destroyed and its replica state appears at the remote place after applying local measurement in the form of linear operators. To see more clearly the entanglement teleportation of two qubits, in this paper the original input state is assumed to be an entangled two-body pure spin-1/2 state $|\varphi_{in}\rangle=\cos(\theta/2)|10\rangle+e^{i\phi}\sin(\theta/2)|01\rangle$ ($0\leqslant\theta\leqslant\pi$ and $0\leqslant\phi\leqslant2\pi$). Here the different values of θ describe all states with different amplitudes, and ϕ are the phase of these states. The output state is given by [10, 19]

$$\rho_{\text{out}}=\sum_{ij=0}^{3}P_{ij}(\sigma_i\otimes\sigma_j)\rho_{\text{in}}(\sigma_i\otimes\sigma_j),\tag{4}$$

where σ_0 is the identity matrix and σ_i ($i=1,2,3$) is the three components of the Pauli matrix. $P_{ij}=\text{Tr}[E^i\rho(T)]\text{tr}[E^j\rho(T)]$, $\sum_{ij}P_{ij}=1$ and ρ_{in} is the density matrix of input state. Here $E^0=|\Psi^-\rangle\langle\Psi^-|$, $E^1=|\Phi^-\rangle\langle\Phi^-|$, $E^2=|\Phi^+\rangle\langle\Phi^+|$, $E^3=|\Psi^+\rangle\langle\Psi^+|$, where $|\Psi^{\pm}\rangle=(1/\sqrt{2})(|01\rangle\pm|10\rangle)$ and $|\Phi^{\pm}\rangle=(1/\sqrt{2})(|00\rangle\pm|11\rangle)$ respectively. After some straightforward algebra, we obtain the output density matrix has the form of

$$\rho(T)=\begin{pmatrix}\alpha&0&0&0\\0&y_1&y_2&0\\0&y_3&y_4&0\\0&0&0&\alpha\end{pmatrix}.$$

These nonzero matrix elements are expressed as following

$$\begin{aligned}\alpha&=(\mu_{+}+\mu_{-})(\omega_1+\omega_2),\\ y_1&=(\mu_{+}+\mu_{-})^2\sin\left(\frac{\theta}{2}\right)^2+(\omega_1+\omega_2)^2\cos\left(\frac{\theta}{2}\right)^2,\\ y_2&=4y^2e^{i\phi}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right),\\ y_3&=4y^2e^{-i\phi}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right),\\ y_4&=(\mu_{+}+\mu_{-})^2\cos\left(\frac{\theta}{2}\right)^2+(\omega_1+\omega_2)^2\sin\left(\frac{\theta}{2}\right)^2.\end{aligned}\tag{5}$$

From (2)–(5) and the definition of the concurrence, the entanglement of ρ_{out} can be expressed as

$$C_{\text{out}} = \max\left\{\frac{C_{\text{in}} e^{\frac{J}{T}} \sinh(\frac{J\eta}{T})^2 - 2\eta^2 \cosh(\frac{B}{T})\cosh(\frac{J\eta}{T})}{\eta^2[e^{-\frac{J}{2T}}\cosh(\frac{B}{T}) + e^{\frac{J}{2T}}\cosh(\frac{J\eta}{T})]^2}, 0\right\}, \tag{6}$$

where $C_{\text{in}} = 2|\sin(\theta/2)\cos(\theta/2)|$ is the concurrence of the input state φ_{in}. From (6), one can note that if $C_{\text{in}} = 0$, the quantity C_{out} is always zero no matter B and b are increased or not. C_{out} is increased with increasing the value of C_{in}. This is due to their linear relationship shown in (6).

Figure 1(a) is a plot of C_{out} as a function of J and B when $b = 0$, which means the two-qubit is exerted the same magnetic fields. And Fig. 1(b) shows the dependence of C_{out} on J and the nonuniform magnetic field b for $B = 0$. At this case, the two-qubit is exerted opposite magnetic fields. For both these two case, we set $C_{\text{in}} = 1$ and $T = 0.1$. From Fig. 1(a), we know that C_{out} is keep the maximal value of one when $B = 0$, and then decreased sharply to zero with increasing B, this is due to the fact that the entanglement of this quantum channel decreases when the magnetic field is increased. One can also note that there is no teleportated entanglement in the region of $J < 0$ for any value of B. However, for the nonuniform field case (Fig. 1(b)), the entanglement C_{out} exists when $J < 0$ if the nonuniform field b is with the region $0.5 < b < 3$. The reason is that the entanglement of our channel system is no longer zero by introducing the nonuniform magnetic field for $J < 0$. In addition, when $J > 0$, the region of existing C_{out} in terms of b is larger than that of Fig. 1(a) case. So the nonuniform field can teleportate entanglement for ferromagnetic chain channel as well as for the antiferromagnetic case.

The teleportated entanglement C_{out} as a function of the temperature T with different values of B or b are shown in Fig. 2. It shows that with increasing the temperature T, the value of C_{out} has a decreasing tendency and finally decreased to zero when the threshold temperature arrived, and the reason is that the entanglement of the teleportation channel is decreased with the increase of T. From the Fig. 2(a) we note that the threshold temperature T_c is decreased with increasing the uniform magnetic field B. But the effect of the nonuniform magnetic field b on the threshold temperature is different, where the threshold temperature T_c is improved with increasing the value of b. We can see it clearly from Fig. 2(b).

Next, we will concentrate on the quality of quantum state transfer. A very simple measure of the quality of such a spin quantum channel is the fidelity. The fidelity between ρ_{out} and ρ_{in} defined by [20]

$$F(\rho_{\text{in}}, \rho_{\text{out}}) = \left\{\text{tr}\left[\sqrt{(\rho_{\text{in}}^{1/2})\rho_{\text{out}}(\rho_{\text{in}}^{1/2})}\right]\right\}^2. \tag{7}$$

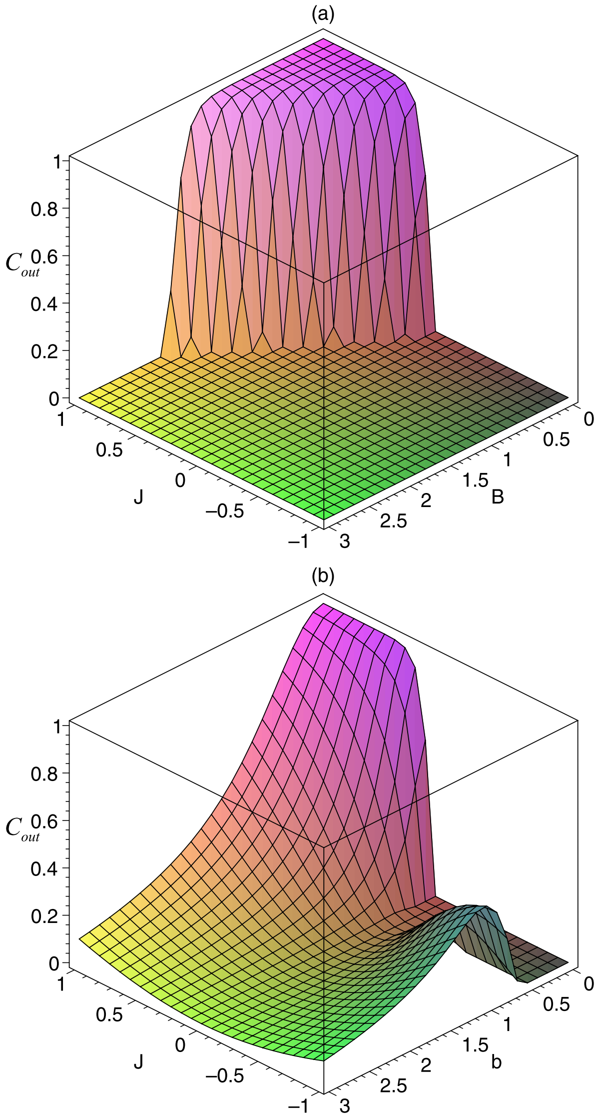
Since the transportated state is a pure state through this spin mixed channel, we are interested in the average fidelity F_a which can be formulated as

$$F_a = \frac{\int_0^{2\pi} d\phi \int_0^{\pi} F\sin(\theta)d\theta}{4\pi}. \tag{8}$$

If this Heisenberg chain under a nonuniform magnetic field as a quantum channel, the analytic expression of average fidelity can be expressed as

$$F_a = \frac{4}{3Z^2}\left[2e^{J\beta}\cosh(J\beta\eta)^2 + e^{-J\beta}\cosh(B\beta)^2 + \frac{1}{\eta^2}e^{J\beta}\sinh(J\beta\eta)^2\right]. \tag{9}$$

Fig. 1 The teleported entanglement C_{out} as a function of the coupling constant J where (**a**) the magnetic field B is changed with $b=0$; (**b**) the nonuniform magnetic field b is changed with $B=0$. We set $C_{\text{in}}=1$ and $T=0.1$



For nonuniform magnetic field case, we will directly numerical plot the figure. We know that in order to transmit $|\varphi_{\text{in}}\rangle$ with better fidelity than any classical communication protocol, we require the value of F_a to be strictly greater than $2/3$, it is about 0.667.

The average fidelity Fa as a function of spin coupling J and B is shown in Fig. 3(a) with $b=0$ at a finite temperature $T=0.1$. It shows that Fa keep a constant 1 if J is large enough and then drops suddenly to zero as a critical B_c is reached when $J>0$. The value of B_c is increased if the interaction of coupling J is enhanced. For $B>B_c$ the fidelity will appeared again and equal to a constant which is smaller than the classical fidelity. For $J<0$, the average fidelity is always smaller than $2/3$. However, if the magnetic fields is opposite, for the ferrimagnetic chain($J<0$), the average fidelity can be larger than $2/3$ which is show in Fig. 3(b). And for $J>0$, the average fidelity Fa is always larger than $2/3$. So, the opposite

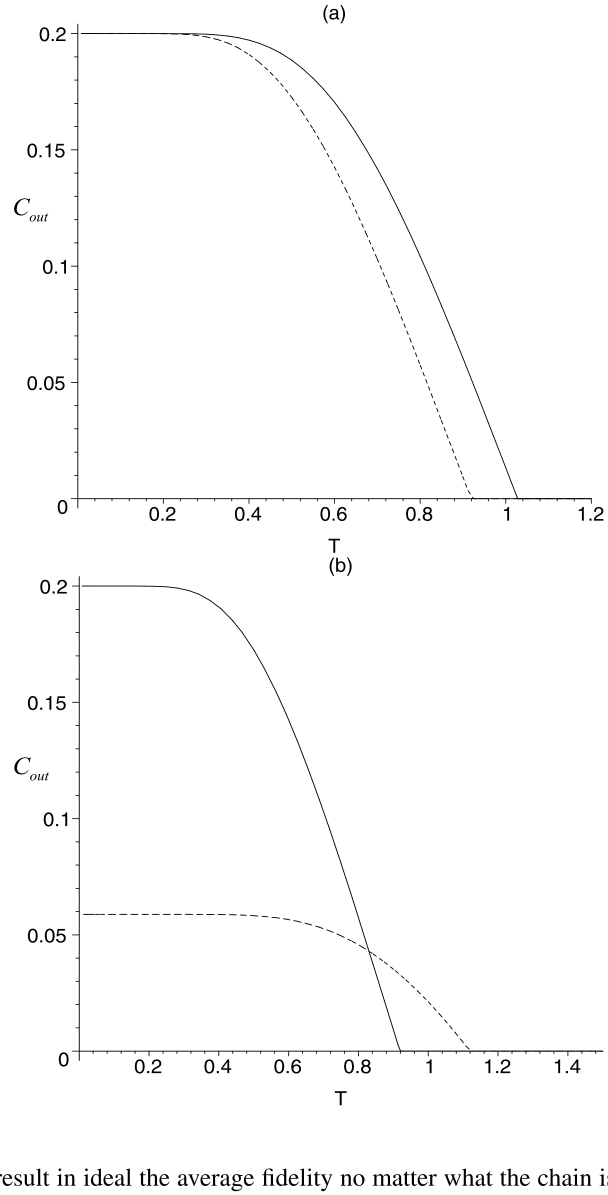


Fig. 2 The teleportated entanglement C_{out} versus the temperature T with different values of B or b. (**a**) For $b = 2$,the solid line and the dotted line correspond to $B = 0.5$, $B = 1$ respectively; (**b**) For $B = 1$, the solid line and the dotted line correspond to $b = 2$, $b = 4$ respectively. For all plot, we set $C_{\text{in}} = 1$ and $J = 1$

direction magnetic fields can result in ideal the average fidelity no matter what the chain is ferromagnetic or antiferromagnetic.

3 Conclusion

In conclusion, it is analyzed the entanglement teleportation of two-qubit pure state via 1D Heisenberg chain under a nonuniform magnetic field. Our study shows that for ferromagnetic chain $J < 0$, a magnetic field in opposite direction can result in an output entanglement C_{out}, while a uniform magnetic field can not do it. When the uniform magnetic field B and the nonuniform magnetic field b coexist, they have different effects on the threshold tem-

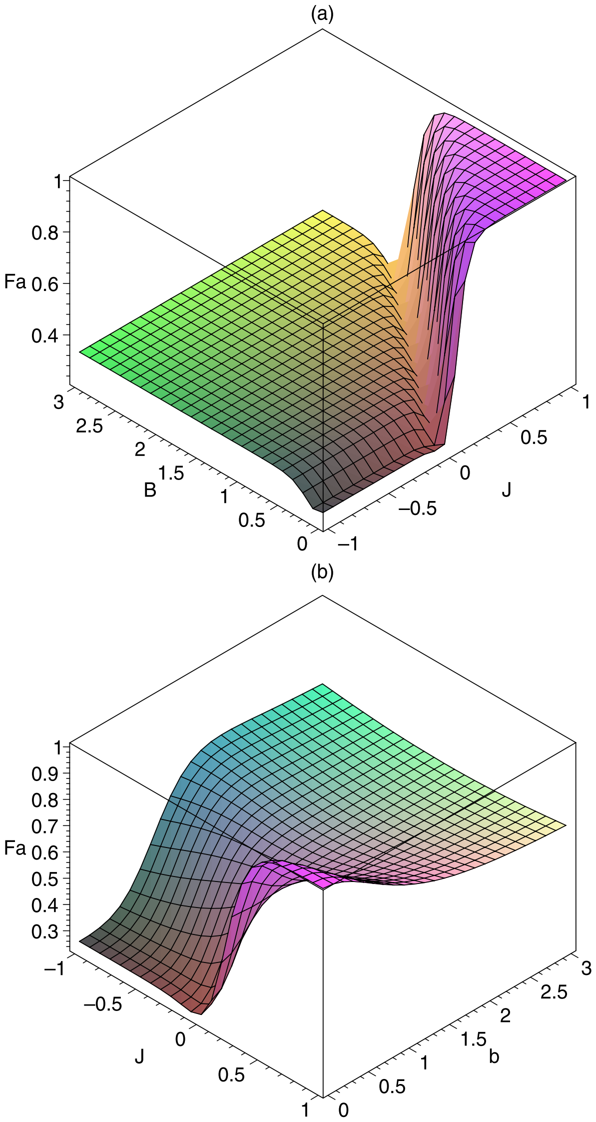


Fig. 3 (**a**) The average fidelity Fa as a function of the magnetic field B and the coupling constant J with $b = 0$. (**b**) The average fidelity Fa versus the nonuniform magnetic field b and J when $B = 0$. where we set a finite temperature $T = 0.1$

perature T_c. For $J = 1$, the larger the value of b is, the larger the threshold temperature T_c is, but the larger the value of B is, the smaller T_c is. We also analyze the average fidelity of the quantum channel system. Our results show that the magnetic field in opposite direction can result in the ideal average fidelity no matter whether the chain is ferromagnetic or antiferromagnetic, while the uniform magnetic field can lead to ideal average fidelity only when $J > 0$.

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